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Generalized Bessel Multiplier on LCA Groups

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Abstract. This paper introduces Bessel admissible families on second countable locally compact abelian (LCA) groups and develops a framework for generalized Bessel multipliers in this setting. Using the associated analysis and synthesis operators, these multipliers are defined and their main properties are investigated. The concept of biorthogonality is further extended to admissible families on LCA groups, and its influence on the structure and stability of the multipliers is analyzed. A key contribution of the study is the examination of the relationship between multipliers on the original space and those on the corresponding fiber space, providing a clearer understanding of their overall structure and interactions.

1. Introduction

The concept of frames in Hilbert spaces was first introduced by Duffin and Schaeffer [10] in 1952 to study some profound issues in nonharmonic Fourier series. A discrete frame is a countable family of elements in a separable Hilbert space which allows stable not necessarily unique (redundant) decompositions of arbitrary elements into expansions of frame element. This concept was generalized by Ali, Antoine and Gazeau [2], to families indexed by an ordered set endowed with a Radon measure [4, 12, 16, 17]. These frames are known as continuous frames. The continuous wavelet transformation and short time Fourier transformation are examples of continuous frames. Bessel multipliers have been introduced by Balazs in [5] for Hilbert space. A multiplier for two Bessel sequences is an operator that combines the analysis operator, a multiplication template with a fixed sequence, called the symbol, and the

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synthesis operator. Moreover, several generalizations of multipliers are proffered [1, 3, 6, 22]. These operators have many applications, such as image and signal processing, quantum mechanics and optimization problems. Also they are using for denoising [19], computational auditory scene analysis [25] and in psychoacoustics[7]. Hence, we refer to [3] for an introduction to the concept of the generalized Bessel multiplier concerning a measure space (Ω, μ) and their properties.When the underlying space is a locally compact abelian (LCA) group G, its algebraic and topological structures enable the use of harmonic analysis tools such as group Fourier analysis, translation operators, and fiberization. These tools provide a richer framework that can facilitate applications including, time-frequency analysis and signal reconstruction in related contexts. Unlike standard continuous frames where elements are arbitrary, our admissible families exploit the translation symmetry of LCA groups, leading to multipliers with specific invariance properties. This motivates us to express the generalized Bessel multiplier with respect to admissible families and to investigate its properties. In what follows we describe the content of the paper. In Section 2, we fixed our notation and review the basic concepts in Hilbert spaces and Fourier analysis on LCA-groups. The most important of these topics are Schatten-von Neumann classes S_P spaces, continuous frame, admissible and fiberization. In Section 3, we recalling the definition of admissible families in the setting of LCA-groups. We then introduce the concept of Bessel admissible families and illustrate it with several examples to aid understanding and note that, when the admissible family is indexed by a countable set of generators and the translation subgroup is discrete, our generalized framework reduces to the classical setting of Gabor and affine-type systems on $L^2(G)$. For more details, the reader is referred to [18]. Generalized Bessel multipliers via admissible families and the concept of biorthogonality for Bessel systems are introduced in Section 3. In Section 4, the fibrization map is defined and for a closed cocompact subgroup H of LCA-group G, and a measure space J; it is shown that the family $\{L_h g_j\}_{h\in H, j\in J}$ associated with admissible system $\{g_j\}_{j\in J} \subset L^2(G)$ is a Bessel family in Hilbert space $L^2(G)$ if and only if the corresponding family of fibers $\{\tau g_j(\omega)\}_{j\in J}$ is a Bessel family in $l^2\left(H^{\perp}\right)$ [16]. Finally, Section 5 presents a concrete application of our theory to signal processing. We demonstrate how Gabor multipliers can be effectively employed for signal denoising on the time-frequency plane $\mathbb{R}^2$, thereby validating the practical utility of the abstract framework developed in previous sections.

2. Notation and Preliminaries

At first we will give only a short review of Hilbert spaces and operators; for details refer to [20, 24, 26]. We will denote the complex Hilbert spaces by $\mathcal{H}$ (respectively $\mathcal{H}_1$,$\mathcal{H}_2$) and their inner product with $\langle .,. \rangle$, which is linear in the first coordinate. Let $B(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all linear and bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. With the operator norm, $\|T\| = \sup_{\|x\|=1} \|Tx\|$, this set is a C^*-algebra and $B(\mathcal{H}, \mathcal{H})$ is abbreviates $B(\mathcal{H})$ and the set of all compact operators on $\mathcal{H}$ denoted by $K(\mathcal{H})$ is a closed two-sided ideal in $B(\mathcal{H})$. The singular values of a compact operator T on $\mathcal{H}$ are the sequence roots of the positive eigenvalues of the operator T^*T, where T^* denotes the adjoint of T. Equivalently, this is the sequence of positive eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$. We always assume, without loss of generality, that the sequence of singular values of T, $\{\lambda_n\}$, is non-increasing. Given $0 < p < \infty$, the schatten p-class of $\mathcal{H}$, denoted $S_p(\mathcal{H})$ or simply S_p, is defined as the space of all compact operators T on $\mathcal{H}$ with the singular value sequence $\{\lambda_n\}$ belong to l^p. It is well known that S_p is a two-sided ideal in $K(\mathcal{H})$. Also, when equipped with

$$\|T\|_{S_p} = \left[\sum_{n=1}^{\infty} \lambda_n^p\right]^{\frac{1}{p}},$$

S_p is a Banach space when $1 \leq p < \infty$ and a complete metric space when $0 < p < 1$. Two specific cases are specifically interesting in operator theory: S_1 is called the trace class and S_2 is the Hilbert-Schmidt class.

One of the main concept of this paper is the Bessel mapping via a measure space. The definition is as follows. Let $\mathcal{H}$ be a complex Hilbert space and (M, μ) be a measure space with a positive measure μ. The space $L^2(M, \mu)$ is defined as the set of all measurable functions $f : M \to \mathbb{F}$ satisfying

$$\int_M |f(m)|^2 \, d\mu(m) < \infty.$$

The mapping $F : M \to \mathcal{H}$ is called a (general) Bessel mapping with bound B for $\mathcal{H}$ with respect to (M, μ), if

(1) F is weakly-measurable, i.e., for all $f \in \mathcal{H}$, $m \to \langle f, F(m) \rangle$ is a measurable function on M;

(2)there exist constants $B > 0$ such that

$$\int_M |\langle f, F(m) \rangle|^2 \, d\mu(m) \leq B \|f\|^2, \quad (f \in \mathcal{H}). \tag{2.1}$$

The mapping F is called a (general) continuous frame for $\mathcal{H}$ with respect to measure space (M,μ) if (2.1) holds and there exist $A>0$ such that

$$A\left\|f\right\|^{2}\leq\left(\int_{M}\left|\left\langle f,F\left(m\right)\right\rangle\right|^{2}d\mu\left(m\right)\right),\quad\forall f\in\mathcal{H}.$$

The constant A,B are called the frame bounds. when $A=B$, F is called a (general) continuous tight frame and the case $A=B=1$, is known as the (general) normalized tight frame or Parseval frame.

Consider a Bessel mapping $F:M\rightarrow\mathcal{H}$. The operator

$$T_{F}:L^{2}\left(M,\mu\right)\rightarrow\mathcal{H}$$

can be defined by

$$\left\langle T_{F}\varphi,f\right\rangle=\int_{M}\varphi\left(m\right)\left\langle F\left(m\right),f\right\rangle d\mu\left(m\right),\quad\left(\varphi\in L^{2}\left(M,\mu\right),f\in\mathcal{H}\right)$$

and is referred to as the synthesis operator. The adjoint operator $T_{F}^{*}:\mathcal{H}\rightarrow L^{2}\left(M,\mu\right)$ acts in the reverse direction and is called the analysis operator, given pointwise by

$$\left(T_{F}^{*}f\right)\left(m\right)=\left\langle f,F\left(m\right)\right\rangle,\quad\left(f\in\mathcal{H},m\in M\right).$$

The frame operator associated with F is defined through composition as $S_{F}:=T_{F}T_{F}^{*}$ which is positive, bounded, and invertible [12]. Moreover, for a Bessel mapping $F:M\rightarrow\mathcal{H}$, a Bessel mapping $F^{d}:M\rightarrow\mathcal{H}$ is called a dual for F if

$$f=\int_{M}\left\langle f,F^{d}\left(m\right)\right\rangle F\left(m\right)d\mu\left(m\right),\quad\left(f\in\mathcal{H}\right).$$

Every continuous frame $F:M\rightarrow\mathcal{H}$ with frame operator S_{F}, has at least one dual frame, namely $S_{F}^{-1}F$ [4] . Continuous frames that possess a unique dual are referred to as continuous Riesz bases (see[4]). We denote $S_{F}^{-1}F$ by $\tilde{F}$.

For the reader's convenience and fixed notations, we review some basic facts from the theory of locally compact abelian groups.

Let G denote a second countable locally compact abelian group (LCA-group) with a left Haar measure dx which is unique up to a positive constant. The dual group of G, that we denoted by $\widehat{G}$. By the pontryagin duality theorem, the dual group of $\widehat{G}$ is isomorphic to G as a topological group i.e $\widehat{\widehat{G}}\cong G$. Moreover, we remind the well-known facts that if G is discrete, then $\widehat{G}$ is compact, and vice versa.

For a subset H of a LCA-group G, we define its annihilator as

$$H^{\perp}=\left\{\omega\in\widehat{G};\omega\left(h\right)=1,\forall h\in H\right\}.$$

The annihilator is a closed subgroup in $\widehat{G}$, and if H is a closed subgroup, then $H^{\perp} \approx \widehat{\left(\frac{G}{H}\right)}$ and $\frac{\widehat{G}}{H^{\perp}} \approx \widehat{H}$.

We use the Weil's formula in our result process. Indeed, if H is a closed subgroup of LCA-group G and $f \in L^1(G)$, then the Haar measure dx, dh and $d(xH)$ can be chosen such that

$$\int_G f(x)\, dx = \int_{\frac{G}{H}} \int_H f(xh)\, dh d(xH),$$

where xH denotes the coset of x in the quotient group $\frac{G}{H}$.

In fact, Weil's formula [23] relates integrable function over G with integrable function on the quotient group G/H when H is a closed subgroup of G.

For more information on harmonic analysis in LCA-groups, is referred [11, 14, 15, 23].

3. Generalized Bessel multiplier

Let G_1, G_2 be two LCA-groups, in this section, we intend to introduce the generalized Bessel multipliers with respect to admissible families $\{g_j\}_{j\in J} \subset L^2(G_1), \{f_j\}_{j\in J} \subset L^2(G_2)$ and investigate their properties. First, we will review the definition of an admissible family, and then to clarify the topic, we will provide an example of this family. Additionally, we will introduce a fundamental definition of a Bessel mapping and present some illustrative examples.

Let G be a second countable locally compact group and J be a countable or uncountable index set. Let $g_j \in L^2(G)$ for $j \in J$, and let H be a closed subgroup in G. The translation invariant system generated by $\{g_j\}_{j\in J}$ with translation along H is the family of functions $\{L_h g_j\}_{h\in H, j\in J}$; where $L_h g_j(x) = g_j\left(h^{-1}x\right)$. If the following fixed assumption holds in the index set J:

(i) The measure space (J, Σ_J, μ_J) is σ-finite,
(ii) $j \mapsto g_j$, $(J, \Sigma_J) \to \left(L^2(G), B_{L^2(G)}\right)$ is measurable,where $B_{L^2(G)}$ is the Borel functions on $L^2(G)$,
(iii) $(j, x) \mapsto g_j(x)$, $(J \times G, \Sigma_J \otimes B_G) \to (\mathbb{C}, B_{\mathbb{C}})$ is measurable.

Then the family $\{g_j\}_{j\in J}$ is called an admissible family for $L^2(G)$ along closed subgroup H (see [16]).

It is worthwhile to mention that for a second countable LCA-group G, a closed subgroup H of G, and $g \in L^2(G)$, the family $\{L_h g_j\}_{h\in H, j\in J}$ is admissible for $L^2(G)$, where $g_j = L_j g$ and J is a subgroup of G or $\widehat{G}$, or $G \times \widehat{G}$.

Definition 3.1. Let G be LCA-group and H be a closed subgroup of G, let J be a measure space and $\{g_j\}_{j\in J} \subset L^2(G)$ be admissible family. The mapping

$$\begin{aligned} \mathfrak{F} : J \times H &\to L^2(G) \\ (j,h) &\mapsto L_h g_j \end{aligned}$$

is called a Bessel mapping if there exist constant $B > 0$ such that

$$\int_{J\times H} |\langle f, L_h g_j\rangle|^2 \, d(h,j) \leq B \|f\|^2, \quad \left(\forall f \in L^2(G)\right). \tag{3.1}$$

In this case the family $\{L_h g_j\}_{h\in H, j\in J}$ is called a Bessel admissible family (system). The mapping $\mathfrak{F}$ is called a continuous frame if (3.1) holds and there exists A > 0 such that $A\|f\|^2 \leq \int_{J\times H} |\langle f, L_h g_j\rangle|^2 \, d(h,j)$, for all $f \in L^2(G)$.The constant A, B are called the frame bounds. when $A = B$, $\mathfrak{F}$ is called a continuous tight frame and the case $A = B = 1$, is known as the normalized tight frame or Parseval frame.

Remark 3.2. It is important to distinguish Definition 3.1 from the (general) continuous Bessel mappings defined in Section 2. While Section 2 deals with arbitrary measurable maps $F : M \to \mathcal{H}$, Definition 3.1 considers families generated by the action of a subgroup H on a base family $\{g_j\}$. This structural assumption ensures that the mapping $(j,h) \mapsto L_h g_j$ is continuous (not just measurable), allowing the use of Haar measure integration techniques. This structure is essential for the results in Section 4, particularly for establishing the translation invariance of the associated Bessel multipliers, which does not hold for (general) continuous frames.

The following two examples provide a clarification for this definition.

Example 3.3. Let $G = \mathbb{Z}, N \in \mathbb{N}, N \geq 2$ and consider the co-compact subgroup $H = N\mathbb{Z}$. Note that $\widehat{G}$ can be defined with half-open unit interval $[0,1)$ under addition module one. To each H we associate N function g_j for $J = \{0, 1, \ldots, N-1\}$. Function g_j is defined by Fourier transform

$$\hat{g}_j = (N-1)^{\frac{1}{2}} N^{\frac{-1}{2}} \chi_{\left[\frac{j}{N}, \frac{(j+1)}{N}\right)},$$

the factor $(N-1)^{\frac{1}{2}}$ is for normalization purpose and not play a role in the calculations. The annihilator of H is given $H^{\perp} = N^{-1}\mathbb{Z} \cap [0,1)$. Note that the number of elements in $H^{\perp}$ is N. We equip both G and H with the counting measure, this implies that the measure on H is the counting measure multiplied by N. The shift invariant system $\{L_h g_j\}_{h\in H, j\in J}$ is

a Parseval frame for $l^2(\mathbb{Z})$. Indeed Note that

$$\sum_{j=0}^{N-1} |\hat{g}_j(\omega)|^2 = (N-1)N^{-1}\chi_{[0,1)}(\omega)$$

for $\omega \in [0,1)$. Using the bound estimates from ([17], Proposition 3.7) we have

$$\begin{aligned} B &= \operatorname*{ess\,sup}_{\omega\in[0,1)} \sum_{j=0}^{N-1} \sum_{\alpha\in H^{\perp}} |\hat{g}_j(\omega)\,\hat{g}_j(\omega+\alpha)| \\ &= \operatorname*{ess\,sup}_{\omega\in[0,1)} \sum_{j=0}^{N-1} |\hat{g}_j(\omega)|^2 \\ &= \operatorname*{ess\,sup}_{\omega\in[0,1)} (N-1)N^{-1}\chi_{[0,1)}(\omega) \\ &< \infty. \end{aligned}$$

Example 3.4. Let $G = \mathbb{R}$ and let $H = \mathbb{Z}$ be a closed subgroup of G. Consider the measure space $J = \mathbb{Z}$ and fix the function $g(x) := \chi_{[0,1]}(x) \in L^2(\mathbb{R})$. For $j \in J$, define modulated functions

$$g_j(x) := \mathcal{M}_j g(x) = e^{2\pi ijx}\chi_{[0,1]}(x),$$

where $\mathcal{M}_j$ denotes modulation by frequency j.

The family $\{L_h g_j\}_{h,j\in\mathbb{Z}} = \left\{e^{-2\pi ijh}\mathcal{M}_j L_h \chi_{[0,1]}\right\}_{h,j\in\mathbb{Z}}$ can be identified (up to unimodular scalar factors) with the lattice Gabor system

$$\mathcal{G}(g,a,b) := \left\{e^{2\pi ibjx} g(x-ah)\right\}_{h,j\in\mathbb{Z}},$$

where the lattice parameters $a = b = 1$. It is well-known that

$$\mathcal{G}\left(\chi_{[0,1]},1,1\right) = \left\{\mathcal{M}_j L_h \chi_{[0,1]}\right\}_{j,h\in\mathbb{Z}}$$

is a Parseval frame (Exercise 11.5 [13]). This implies that for all $f \in L^2(\mathbb{R})$

$$\sum_{j,h\in\mathbb{Z}} |\langle f, L_h g_j\rangle|^2 = \|f\|^2,$$

which satisfies the Bessel condition with constant $B = 1$ and therefore $\{L_h g_j\}_{j,h\in\mathbb{Z}}$ is a Bessel admissible family (Note that The phase factors $e^{-2\pi ijh}$ have modulus one and therefore do not affect the frame or Bessel properties).

Let $\{g_j\}_{j\in J}$ be admissible family for $L^2(G)$ and H be closed subgroup of G, $\mathfrak{F}$ be a Bessel system from $J \times H$ to $L^2(G)$. Then the operator

$T_{\mathfrak{F}} : L^2(J \times H) \to L^2(G)$ defined by

$$\langle T_{\mathfrak{F}}\varphi, f\rangle = \int_J \int_H \varphi(j,h)\langle L_h g_j, f\rangle \, dhdj, \quad \left(\varphi \in L^2(J\times H), f \in L^2(G)\right)$$

is well-defined, linear and bounded. Indeed

$$\begin{aligned} |\langle T_{\mathfrak{F}}\varphi, f\rangle| &\leq \int_J \int_H |\varphi(j,h)\langle L_h g_j, f\rangle| \, dhdj \\ &\leq \left(\int_J \int_H |\varphi(j,h)|^2 \, dhdj\right)^{\frac{1}{2}} \left(\int_J \int_H |\langle L_h g_j, f\rangle|^2 \, dhdj\right)^{\frac{1}{2}} \\ &\leq \|\varphi\|_2 \sqrt{B}\, \|f\|, \end{aligned}$$

and

$$\begin{aligned} T_{\mathfrak{F}}^* : L^2(G) &\to L^2(J\times H) \\ \left(T_{\mathfrak{F}}^* f\right)(i,t) &= \langle f, L_t g_i\rangle. \end{aligned}$$

In fact, for $\varphi \in L^2(J\times H)$ we have,

$$\begin{aligned} \left\langle T_{\mathfrak{F}}^* f, \varphi\right\rangle &= \int_J \int_H T_{\mathfrak{F}}^* f(j,h)\,\overline{\varphi(j,h)} dhdj \\ &= \int_J \int_H \langle f, L_h g_j\rangle\, \overline{\varphi(j,h)} dhdj \\ &= \overline{\langle T_{\mathfrak{F}}\varphi, f\rangle} \end{aligned}$$

so $\left\langle T_{\mathfrak{F}}^* f, \varphi\right\rangle = \langle f, T_{\mathfrak{F}}\varphi\rangle$.

The operator $T_{\mathfrak{F}}$ is called synthesis operator and $T_{\mathfrak{F}}^*$ is called analysis operator of $\mathfrak{F}$.

Proposition 3.5. *Let G_1 and G_2 be two LCA-groups and H_1, H_2 be two closed subgroups of G_1,G_2 respectively. Also*

$$\{g_j\}_{j\in J} \subset L^2(G_1), \qquad \{f_j\}_{j\in J} \subset L^2(G_2)$$

be two admissible families with translation along H_1 and H_2, respectively. Assume that

$$\mathfrak{F}_1 : J_1 \times H_1 \to L^2(G_1); (j,h_1) \mapsto L_{h_1} g_j$$

and

$$\mathfrak{F}_2 : J_2 \times H_2 \to L^2(G_2); (j,h_2) \mapsto L_{h_2} f_j$$

are Bessel mappings with bounds B_1,B_2 and

$$u : L^2(J_1 \times H_1) \to L^2(J_2 \times H_2)$$

is a bounded linear operator. Then

$$M_{u,\mathfrak{F}_1,\mathfrak{F}_2} := T_{\mathfrak{F}_2} \circ u \circ T_{\mathfrak{F}_1}^* : L^2(G_1) \to L^2(G_2)$$

is well-defined bounded operator, and

$$\|M_{u,\mathfrak{F}_1,\mathfrak{F}_2}\|_{op} \leq \sqrt{B_1 B_2}\,\|u\|_{op}.$$

Moreover $M^*_{u,\mathfrak{F}_1,\mathfrak{F}_2} = M_{u^*,\mathfrak{F}_2,\mathfrak{F}_1}$.

Proof. The proof is straightforward □

The operator $M_{u,\mathfrak{F}_1,\mathfrak{F}_2}$, introduced in proposition 3.5 is said the generalized Bessel multiplier respect to admissible families

$$\{g_j\}_{j\in J} \subset L^2(G_1), \qquad \{f_j\}_{j\in J} \subset L^2(G_2)$$

or generalized Bessel multiplier associated with the $L_{h_1} g_j$ and $L_{h_2} f_j$. We denote $M_{u,\mathfrak{F}_1}$ for $M_{u,\mathfrak{F}_1,\mathfrak{F}_1}$.

The next two propositions explore operator-theoretic properties of generalized Bessel multipliers – such as compactness, Schatten p-class membership, and positivity– arising from admissible families with translation structures on LCA-groups.

Proposition 3.6. *Let G_1, G_2 be two LCA-groups and H_1, H_2 be two closed subgroups of G_1, G_2 respectively. Also*

$$\{g_j\}_{j\in J} \subset L^2(G_1), \qquad \{f_j\}_{j\in J} \subset L^2(G_2)$$

be two admissible families with translation along H_1 and H_2, respectively. Suppose that $\mathfrak{F}_1 : J_1 \times H_1 \to L^2(G_1), \mathfrak{F}_2 : J_2 \times H_2 \to L^2(G_2)$ are Bessel mappings with bounds B_1, B_2, respectively. If $u : L^2(J_1 \times H_1) \to L^2(J_2 \times H_2)$ is compact operator then $M_{u,\mathfrak{F}_1,\mathfrak{F}_2} : L^2(G_1) \to L^2(G_2)$ is compact operator.

Proof. The fact that the set of all compact operators is a closed two-sided ideal of bounded linear operators, shows that for a given compact operator

$$u \in \left(L^2(J_1 \times H_1),\, L^2(J_2 \times H_2)\right),$$

it follows that

$$M_{u,\mathfrak{F}_1,\mathfrak{F}_2} = T_{\mathfrak{F}_2} \circ u \circ T^*_{\mathfrak{F}_1}$$

is also a compact operator. □

Proposition 3.7. *Let G_1, G_2 be two LCA-groups, H_1, H_2 be two closed subgroup of G_1, G_2 respectively, $\{g_j\}_{j\in J} \subset L^2(G_1), \{f_j\}_{j\in J} \subset L^2(G_2)$ be two admissible families. Assume that $\mathfrak{F}_1, \mathfrak{F}_2$ are Bessel systems with bounds B_1 and B_2. If $u \in S_p\left(L^2(J_1 \times H_1), L^2(J_2 \times H_2)\right)$ then $M_{u,\mathfrak{F}_1,\mathfrak{F}_2} \in S_p\left(L^2(G_1), L^2(G_2)\right)$ with*

$$\|M_{u,\mathfrak{F}_1,\mathfrak{F}_2}\|_{S_p} \leq \sqrt{B_1 B_2}\,\|u\|_{S_p}.$$

Moreover if $u \in B\left(L^2(J_1 \times H_1)\right)$ is positive operator then $M_{u,\mathfrak{F}_1} \in L^2(G_1)$ is also positive.

Proof. Since the notion of Schatten p-class is extended to operators between different Hilbert spaces, and due to the fact that the Schatten p-class is stable under composition with bounded operators, it follows that for Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ and operators $\mathcal{U}_1 \in B(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{U}_2 \in B(\mathcal{H}_4, \mathcal{H}_3)$ and $\mathcal{U}_3 \in S_p(\mathcal{H}_3, \mathcal{H}_1)$, we get $\mathcal{U}_1\mathcal{U}_3 \in S_p(\mathcal{H}_3, \mathcal{H}_2)$ and $\mathcal{U}_3\mathcal{U}_2 \in S_p(\mathcal{H}_4, \mathcal{H}_1)$. Furthermore, $\|\mathcal{U}_1\mathcal{U}_3\|_{S_p} \leq \|\mathcal{U}_3\|_{S_p} \|\mathcal{U}_1\|$ and $\|\mathcal{U}_3\mathcal{U}_2\|_{S_p} \leq \|\mathcal{U}_3\|_{S_p} \|\mathcal{U}_2\|$.(see [21]). Therefore since

$$u \in S_p\left(L^2(J_1 \times H_1), L^2(J_2 \times H_2)\right)$$

implies that $M_{u,\mathfrak{F}_1,\mathfrak{F}_2} \in S_p\left(L^2(G_1), L^2(G_2)\right)$. Also the fact that $\varphi \in L^2(J_1 \times H_1)$ is positive, implies that for each $f \in L^2(G_1)$,

$$\langle f, M_{u,\mathfrak{F}_1}(f)\rangle = \left\langle T^*_{\mathfrak{F}_1}(f), uT^*_{\mathfrak{F}_1}(f)\right\rangle \geq 0. \qquad \square$$

The following proposition highlights structural properties of generalized Bessel multiplier $M_{u,\mathfrak{F}_1,\mathfrak{F}_2}$ associated with admissible families equipped with translation structures on LCA-groups.

Proposition 3.8. *Let G_1 and G_2 be two LCA-groups, and let H_1 and H_2 be closed subgroups of G_1 and G_2, respectively. Also $\{g_j\}_{j\in J} \subset L^2(G_1), \{f_j\}_{j\in J} \subset L^2(G_2)$ be two admissible families with translation along H_1 and H_2, respectively. Assume that*

$$\mathfrak{F}_1 : J_1 \times H_1 \to L^2(G_1); (j,h) \mapsto L_h g_j$$

and

$$\mathfrak{F}_2 : J_2 \times H_2 \to L^2(G_2); (j,h) \mapsto L_h f_j$$

are Bessel mappings with bounds B_1,B_2, respectively and

$$u : L^2(J_1 \times H_1) \to L^2(J_2 \times H_2)$$

is a bounded linear operator. Then the following holds.

(i) *The mapping $\mathfrak{F}_1$ is a continuous frame for Hilbert space $L^2(G_1)$ whenever $M_{u,\mathfrak{F}_1,\mathfrak{F}_2} : L^2(G_1) \to L^2(G_2)$ is bounded, injective, and has a closed range.*

(ii) *The mapping $\mathfrak{F}_1, \mathfrak{F}_2$ are both continuous frames whenever there is $A > 0$ such that for any f in $L^2(G_1)$; $A\|f\|^2 \leq |\langle M_{u,\mathfrak{F}_1,\mathfrak{F}_2} f, f\rangle|$.*

Proof. (1) By the fact that injective closed range bounded linear operators are bounded below, we may assume $M_{u,\mathfrak{F}_1,\mathfrak{F}_2}$ is bounded below. So, there exist $g \in L^2(G_2)$; $\|g\| = 1$ and $c\|f\| \leq |\langle M_{u,\mathfrak{F}_1,\mathfrak{F}_2}(f), g\rangle|$. Then, we have,

$$\begin{aligned} c\|f\| &\leq |\langle M_{u,\mathfrak{F}_1,\mathfrak{F}_2}(f), g\rangle| \\ &= \left|\left\langle T_{\mathfrak{F}_2} \circ u \circ T^*_{\mathfrak{F}_1}(f), g\right\rangle\right| \\ &= \left|\left\langle T^*_{\mathfrak{F}_1}(f), u^* \circ T^*_{\mathfrak{F}_2}(g)\right\rangle\right| \end{aligned}$$

$$\leq \left\|u^* \circ T^*_{\mathfrak{F}_2}(g)\right\|_2 \left\|T^*_{\mathfrak{F}_1}(f)\right\|_2$$
$$\leq \|u\| \|g\| \sqrt{B_2} \left(\int_{J\times H} |\langle f, L_h g_j\rangle|^2 d(j,h)\right)^{\frac{1}{2}}$$

hence $\frac{c^2}{B_2\|u\|^2}\|f\|^2 \leq \int_{J\times H} |\langle f, L_h g_j\rangle|^2 d(j,h)$. Since $\mathfrak{F}_1$ is a Bessel mapping. Therefore $\mathfrak{F}_1$ is a continuous frame.

To proving (2), note that

$$A\|f\|^2 \leq |\langle M_{u,\mathfrak{F}_1,\mathfrak{F}_2}(f), f\rangle|$$
$$\leq \|M_{u,\mathfrak{F}_1,\mathfrak{F}_2}(f)\| \|f\|,$$

for every $f \in L^2(G_1)$. This implies that $M_{u,\mathfrak{F}_1,\mathfrak{F}_2}$ is an injective closed range bounded linear operator, so by part (1), $\mathfrak{F}_1$ is a continuous frame, and by a similar argument, one can get

$$A\|f\|^2 \leq \|M_{u^*,\mathfrak{F}_2,\mathfrak{F}_1}(f)\| \|f\|.$$

Hence $M_{u^*,\mathfrak{F}_2,\mathfrak{F}_1}$ is also bounded below operators and so $\mathfrak{F}_2$ is continuous frames. □

Building upon Definition 3.10 from our previous work [3], which introduced the notion of biorthogonality for Bessel mappings in the setting of measure spaces, we now extend this concept within the framework of admissible families over LCA-groups with translation along closed subgroups.

Definition 3.9. Let G be a LCA-group, H be a closed subgroup of G and J be a measure space, $\{g_j\}_{j\in J}, \{f_j\}_{j\in J} \subset L^2(G)$ be admissible families with translation along H_1 and H_2, respectively. Also; suppose that $\mathfrak{F}_1, \mathfrak{F}_2 : J\times H \to L^2(G)$ are Bessel mappings. Two Bessel mappings $\mathfrak{F}_1, \mathfrak{F}_2$ are said *biorthogonal* if for any finite measurable subset Δ of $J\times H$,

$$\left\langle \int_\Delta L_h g_j d(h,j), L_t f_i \right\rangle = \begin{cases} 1 & (t,i) \in \Delta \quad a.e. \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that for $\varphi \in L^2(J\times H)$, if $\mathfrak{F}_1, \mathfrak{F}_2$ are biorthogonal then

$$\left\langle \int_{J\times H} \varphi(h,j) L_h g_j d(h,j), L_t f_i \right\rangle = \begin{cases} \varphi(i,t) & (i,t) \in \operatorname{supp}\varphi \quad a.e. \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.10. Definition 3.9 presents a group-adapted version of the classical notion of biorthogonality introduced in [3]. While the standard definition in [3] applies to arbitrary measurable families on a general measure space (Ω, μ), Definition 3.9 is specifically tailored for Bessel admissible families on LCA groups. The key distinction is that Definition 3.9 exploits the closed subgroup structure (H) and translation operators

(L_h). This structural adaptation allows us to derive biorthogonal relations that are invariant under group actions, which is essential for the theory of Bessel multipliers developed in Section 4. Furthermore, the fiberization map allows the biorthogonality condition in Definition 3.9 to be reduced to simpler conditions on the base families $\{g_j\}$ and $\{f_j\}$ in the frequency domain (or fiber space). This reduction significantly simplifies the analysis compared to the abstract measure-theoretic setting of [3].

In the following, we investigate properties of generalized Bessel multipliers related to biorthogonal Bessel mappings arising from admissible families with translation along closed subgroups of LCA-groups, as established in Theorem 3.11.

Theorem 3.11. *Let G_1, G_2 be two LCA-groups and H_1, H_2 be closed subgroups of G_1, G_2 respectively. Assume that $\{\gamma_j\}_{j\in J}, \{f_j\}_{j\in J} \subset L^2(G_1)$ are admissible families with translation along H_1 and $\mathfrak{F}_1, \mathfrak{F}_2 : J_1 \times H_1 \to L^2(G_1)$ are Bessel systems. Also $\{k_j\}_{j\in J}, \{g_j\}_{j\in J} \subset L^2(G_2)$ are admissible families with translation along H_2 and $\mathfrak{F}_3, \mathfrak{F}_4 : J_2 \times H_2 \to L^2(G_2)$ are Bessel systems such that $\mathfrak{F}_1, \mathfrak{F}_2$ are biorthogonal and $u_1 : L^2(J_1 \times H_1) \to L^2(J_2 \times H_2), u_2 : L^2(J_2 \times H_2) \to L^2(J_1 \times H_1)$ are arbitrary nonzero bounded, linear operator. Then*

$$M_{u_1,\mathfrak{F}_1,\mathfrak{F}_3} \circ M_{u_2,\mathfrak{F}_4,\mathfrak{F}_2} = M_{u_1 \circ u_2,\mathfrak{F}_4,\mathfrak{F}_3}$$

Proof. It is sufficient to show that $T^*_{\mathfrak{F}_1} \circ T_{\mathfrak{F}_2} = I$. To do so for $\varphi \in L^2(J_1 \times H_1)$, we have

$$\begin{aligned}\left(T^*_{\mathfrak{F}_1} \circ T_{\mathfrak{F}_2}(\varphi)\right)(i,t) &= \langle T_{\mathfrak{F}_2}\varphi, L_t f_i\rangle \left\langle \int_{J_1\times H_1} \varphi(h,j)\, L_h\gamma_j d(h,j), L_t f_i \right\rangle \\ &= \varphi(i,t) \quad a.e. \quad (i,t) \in \text{supp}\varphi. \end{aligned}$$

□

4. Relation Between the Multiplier of the Left Translation and the Fiberization Map

Before outlining the main objective of this section, it is important to review the definition of a fiberization map along with one of its key properties. Let G be a LCA-group and H be a closed co-compact subgroup of G and $\Omega \subset \widehat{G}$ be a Borel section of $H^\perp$ in $\widehat{G}$ (i.e A Borel section or a fundamental domain for H is a Borel measurable subset $D \subset G$ such that every $x \in G$ can be uniquely written in the form $x = hd$ for some $h \in H$ and $d \in D$).

Following the approach in [8] the fiberization mapping is defined as

$$\tau : L^2(G) \to L^2\left(\Omega, l^2\left(H^\perp\right)\right), \qquad \tau f(\omega) = \left\{\hat{f}(\omega\alpha)\right\}_{\alpha\in H^\perp}, \quad \omega \in \Omega;$$

the inner product in $L^2\left(\Omega, l^2\left(H^\perp\right)\right)$ is defined in the obvious manner. As established in [8, 9], the mapping τ is an isometric isomorphism. With the assumption above, for $f \in L^2(G)$ and $h \in H$ one can easily check

$$\tau L_h f(\omega) = \overline{\omega}(h)\, \tau f(\omega), \quad (\omega \in \Omega).$$

In this section, we introduce generalized Bessel multipliers associated with systems in $L^2(G)$ and $l^2\left(H^\perp\right)$. In light of Theorem 3.1 ([16]), which establishes the equivalence between the Bessel (or frame) property of the system $\{L_h g_j\}_{h\in H, j\in J}$ and that of the family $\{\tau g_j(\omega)\}_{j\in J}$ for almost every $\omega \in \Omega$, where G is a LCA-group and H is a closed, co-compact subgroup of G, we consider Bessel families in both domains. Accordingly, we define the generalized Bessel multipliers $M_{u,\{L_h g_j\}_{h\in H, j\in J}}$ and $M_{v,\left\{\tau_{g_j}(\omega)\right\}_{j\in J}}$ for $L^2(G)$ and $l^2\left(H^\perp\right)$, respectively, where J is a measure space (see Figure 2).

We start our work with the following useful proposition.

Proposition 4.1. *Let G be LCA-group and H be closed co-compact subgroup of G. Also $\Omega \subset \widehat{G}$ be a Borel section of $H^\perp$ in $\widehat{G}$ and τ be the fiberization map. For arbitrary $h \in H$, and fixed $\omega \in \Omega$, the mapping $\lambda_{\omega,h} : L^2(G) \to l^2\left(H^\perp\right)$ by*

$$\lambda_{\omega,h}(f) = \omega(h)\, \tau f(\omega),$$

is an isometric bounded linear operator.

For proving Proposition 4.1 the following Lemma is needed. Also, we recall that for a closed subgroup H of LCA-group G, if $\rho : C_c(G) \to C_c\left(\frac{G}{H}\right)$ defined by $\rho f(xH) = \int_H f(xh)\, dh$, then its Fourier transform $\widehat{\rho f} = \widehat{f}|_{H^\perp}$. Moreover, if the Haar measures on H and $H^\perp$ are suitably normalized then for $\widehat{f}|_{H^\perp} \in L^1\left(\widehat{G}\right)$ one get

$$\int_H f(xh)\, dh = \int_{H^\perp} \widehat{f}(\xi)\, \xi(x)\, d\xi. \tag{4.1}$$

Lemma 4.2. *Let G be LCA-group and H be closed subgroup of G and $\Omega \subset \widehat{G}$ be a Borel section of $H^\perp$ in $\widehat{G}$. Also let the mapping $\rho : C_c(G) \to C_c\left(\frac{G}{H}\right)$ defined as $\rho f(xH) = \int_H f(xh)\, dh$. Then for every $f \in L^2(G)$*

$$\widehat{f}(\omega\alpha) = \widehat{(\rho(\overline{w} f))}(\alpha), \quad \left(\omega \in \Omega, \alpha \in H^\perp\right).$$

Proof. We start the proof with the concept of Fourier transform. For $f \in L^1 \cap L^2(G)$, we have

$$\widehat{f}(\omega\alpha) = \int_G f(x)\, \overline{(\omega\alpha)}(x)\, dx$$

$$
\begin{aligned}
&= \int_{\frac{G}{H}} \int_{H} \left(f.\overline{\omega\alpha}\right)(xh)\, dhd\,(xH) \\
&= \int_{\frac{G}{H}} \int_{H} \left(f.\overline{\omega}\right)(xh)\, \overline{\alpha}\,(xh)\, dhd\,(xH) \\
&= \int_{\frac{G}{H}} \overline{\alpha}\,(x) \int_{H} \left(f.\overline{\omega}\right)(xh)\, dhd\,(xH) \\
&= \int_{\frac{G}{H}} \left(\rho\left(f\overline{\omega}\right)\right)(xH)\, \overline{\alpha}\,(xH)\, d\,(xH) \\
&= \left(\rho\left(\overline{\omega}f\right)\right)\widehat{\ }(\alpha)\,.
\end{aligned}
$$

Now by the density of $L^1 \cap L^2(G)$ in $L^2(G)$, we are done. □

Proof of Proposition 4.1. It is enough to show that, $\langle \lambda_{\omega,h} f, \lambda_{\omega,h} g\rangle_{l^2(H^\perp)} = \langle f, g\rangle_{L^2(G)}$ for an $f, g \in L^2(G)$. Indeed, by using the above mentioned fact and Lemma 4.2, we get

$$
\begin{aligned}
\langle \lambda_{\omega,h} f, \lambda_{\omega,h} g\rangle_{l^2\left(H^\perp\right)} &= \langle \omega(h)\,\tau f(\omega)\,, \omega(h)\,\tau g(\omega)\rangle_{l^2\left(H^\perp\right)} \\
&= \langle \tau f(\omega)\,, \tau g(\omega)\rangle_{l^2\left(H^\perp\right)} \\
&= \sum_{\alpha\in H^\perp} \left(\tau f(\omega)\right)(\alpha)\, \overline{\tau g(\omega)}\,(\alpha) \\
&= \sum_{\alpha\in H^\perp} \widehat{f}\,(\omega\alpha)\, \overline{\widehat{g}\,(\omega\alpha)} \\
&= \sum_{\alpha\in H^\perp} \left(\rho\left(\overline{\omega}f\right)\right)\widehat{\ }(\alpha)\, \overline{\left(\rho\left(\overline{\omega}g\right)\right)\widehat{\ }}\,(\alpha) \\
&= \left\langle \left(\rho\left(\overline{\omega}f\right)\right)\widehat{\ }, \left(\rho\left(\overline{\omega}g\right)\right)\widehat{\ } \right\rangle_{l^2\left(H^\perp\right)} \\
&= \left\langle \left(\overline{\omega}f\right)\widehat{\ }, \left(\overline{\omega}g\right)\widehat{\ } \right\rangle_{L^2\left(\widehat{G}\right)} \\
&= \langle \overline{\omega}f, \overline{\omega}g\rangle_{L^2(G)} \\
&= \langle f, g\rangle_{L^2(G)}\,.
\end{aligned}
$$

□

For the next proposition, we need the following notations. Let G be a second countable LCA-group and H be a closed co-compact subgroup of G and J be a measure space, let $\Omega \subset \widehat{G}$ be a Borel section of $H^\perp$ in $\widehat{G}$, $\{g_j\}_{j\in J}$ be an admissible family with along H. For $\omega \in \Omega$ and the Bessel system $\{\tau g_j(\omega)\}_{j\in J} \subseteq l^2\left(H^\perp\right)$ the analysis operator $L_\omega : l^2\left(H^\perp\right) \to L^2(J)$ is given by

$$
\left(L_\omega c\right)(j) = \langle c, \tau g_j(\omega)\rangle_{l^2\left(H^\perp\right)}\,.
$$

Then the synthesis operator $L_\omega^* : L^2(J) \to l^2\left(H^\perp\right)$ is given by

$$L_\omega^* f = \left\{ \int_J f(j)\, \hat{g}_j(\omega\alpha)\, d\mu(j) \right\}_{\alpha \in H^\perp}, \quad \left(f \in L^2(J)\right).$$

(for more details see [16]).

Proposition 4.3. *Let G be a LCA-group and $H \subset G$ closed co-compact subgroup of G, J is a measure space. Also let $\{g_j\}_{j\in J} \subset L^2(G)$ be an admissible family, $\mathfrak{F} = \{L_h g_j\}_{h\in H, j\in J}$, and $\{\tau g_j(\omega)\}_{j\in J}$ be Bessel systems for $L^2(G), l^2(H^\perp)$, respectively.*

Fix an element $h \in H$, and define the mapping $\Gamma : L^2(J \times H) \to L^2(J)$ by

$$(\Gamma f)(j) = \int_H f(j,h)\, dh,$$

which is well-defined. For arbitrary nonzero bounded linear operator u on $L^2(J \times H)$ there is nonzero bounded linear operator v on $L^2(J)$ such that $v = \Gamma \circ u \circ \Gamma^*$.

Then we have,

(i) $\Gamma \circ T_{\mathfrak{F}}^* = L_\omega \circ \lambda_{\omega,h}$
(ii) $T_{\mathfrak{F}} \circ \Gamma^* = \lambda_{\omega,h}^* \circ L_\omega^*$
(iii) $\Gamma \circ u = v \circ \Gamma$

and therefore the following diagram commutes.

$$\begin{array}{ccccccc}
L^2(G) & \xrightarrow{T_{\mathfrak{F}}^*} & L^2(J\times H) & \xrightarrow{u} & L^2(J\times H) & \xrightarrow{T_{\mathfrak{F}}} & L^2(G) \\
\downarrow{\scriptstyle \lambda_{\omega,h}} & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \lambda_{\omega,h}} \\
\ell^2\left(H^\perp\right) & \xrightarrow{L_\omega} & L^2(J) & \xrightarrow{v} & L^2(J) & \xrightarrow{L_\omega^*} & \ell^2\left(H^\perp\right)
\end{array}$$

FIGURE 1.

Proof. According to the assumptions, we show that the above relations hold.

(i) By using the definition of the mapping Γ, for $f \in L^2(G)$, we have

$$\begin{aligned}
\left(\Gamma o T_{\mathfrak{F}}^*(f)\right)(j) &= \int_H T_{\mathfrak{F}}^* f(j,h)\, dh \\
&= \int_H \langle f, L_h g_j \rangle dh.
\end{aligned}$$

On the other hand, using Lemma 4.2 and Plancherel theorem, we have,

$$\begin{aligned}
L_\omega\left(\lambda_{\omega,h}\left(f\right)\right)\left(j\right) &= \left\langle \lambda_{\omega,h} f, \tau g_j\left(\omega\right)\right\rangle_{l^2\left(H^\perp\right)} \\
&= \left\langle \omega\left(h\right)\tau f\left(\omega\right), \tau g_j\left(\omega\right)\right\rangle_{l^2\left(H^\perp\right)} \\
&= \left\langle \omega\left(h\right)\tau f\left(\omega\right), w\left(h\right)\tau L_h g_j\left(\omega\right)\right\rangle_{l^2\left(H^\perp\right)} \\
&= \left\langle \tau f\left(\omega\right), \tau L_h g_j\left(\omega\right)\right\rangle_{l^2\left(H^\perp\right)} \\
&= \sum_{\alpha\in H^\perp} \left(\tau f\left(\omega\right)\right)\left(\alpha\right)\overline{\left(\tau L_h g_j\left(\omega\right)\right)}\left(\alpha\right) \\
&= \sum_{\alpha\in H^\perp} \widehat{f}\left(\omega\alpha\right)\overline{\widehat{L_h g_j}\left(\omega\alpha\right)} \\
&= \left\langle L_{\overline{\omega}}\widehat{f}, L_{\overline{\omega}}\widehat{L_h g_j}\right\rangle_{l^2\left(H^\perp\right)} \\
&= \left\langle \widehat{f}, \widehat{L_h g_j}\right\rangle_{l^2\left(H^\perp\right)} \\
&= \sum_{\alpha\in H^\perp} \widehat{f}\left(\alpha\right)\overline{\widehat{L_h g_j}}\left(\alpha\right) \\
&= \sum_{\alpha\in H^\perp} \widehat{f * \overline{L_h g_j}}\left(\alpha\right).
\end{aligned}$$

By taking $x = e_G$ in (4.1) we get;

$$\begin{aligned}
L_\omega\left(\lambda_{\omega,h}\left(f\right)\right)\left(j\right) &= \int_H f * \overline{L_h g_j}\left(h'\right) dh' \\
&= \int_H \left\langle f, L_{h'} L_h g_j\right\rangle dh' \\
&= \int_H \left\langle f, L_{h'} g_j\right\rangle dh'.
\end{aligned}$$

Hence the result is obtained.

(ii) By part(1), $L_\omega \circ \lambda_{\omega,h} = \Gamma \circ T_{\mathfrak{F}}^*$ so $\left(L_\omega \circ \lambda_{\omega,h}\right)^* = \left(\Gamma \circ T_{\mathfrak{F}}^*\right)^*$ thus $T_{\mathfrak{F}} \circ \Gamma^* = \lambda_{\omega,h}^* \circ L_\omega^*$.

(iii) First note that $\Gamma^* : L^2\left(J\right) \to L^2\left(J \times H\right)$ is given by $\Gamma^*\left(\varphi\right)\left(j,h\right) = \varphi\left(j\right)$ where $\varphi \in L^2\left(J\right)$. Then for any $f \in L^2\left(J \times H\right)$; we have

$$\begin{aligned}
\left\langle \Gamma^*\varphi, f\right\rangle_{L^2(J\times H)} &= \left\langle \varphi, \Gamma f\right\rangle_{L^2(J)} \\
&= \int_J \varphi\left(j\right)\overline{\left(\Gamma f\right)}\left(j\right) dj
\end{aligned}$$

$$= \int_J \varphi(j) \int_H \overline{f(j,h)} dh dj$$
$$= \int_J \int_H \Gamma^*(\varphi)(j,h) \overline{f(j,h)} dh dj.$$

Therefore, we conclude that $\Gamma \circ \Gamma^* = \mathrm{Id}_{L^2(J)}$. Now, since u is arbitrary nonzero linear, consider $v = \Gamma \circ u \circ \Gamma^*$. Then $\Gamma \circ u = v \circ \Gamma$. □

In the next theorem we abbreviate $M_{u,\{L_h g_j\}_{h\in H, j\in J}}$ and $M_{v,\{\tau g_j(\omega)\}_{j\in J}}$ by M_u and M_v, respectively. For the generalized Bessel multiplier we can prove the following properties:

Theorem 4.4. *Let G be LCA-group and $H \subset G$ closed co-compact subgroup of G and J be a measure space. Also let $\{g_j\}_{j\in J} \subset L^2(G)$ is admissible family with translation along H. Fix an element $h \in H$, and define the mapping $\Gamma : L^2(J \times H) \to L^2(J)$ by*

$$(\Gamma f)(j) = \int_H f(j,h)\, dh,$$

which is well-defined. Assume for arbitrary nonzero bounded linear operator u on $L^2(J \times H)$ there is nonzero bounded linear operator v on $L^2(J)$ such that $v = \Gamma \circ u \circ \Gamma^*$. Then the following diagram is commutative.

$$\begin{array}{ccc} L^2(G) & \xrightarrow{M_u} & L^2(G) \\ \Big\downarrow{\scriptstyle \lambda_{\omega,h}} & & \Big\downarrow{\scriptstyle \lambda_{\omega,h}} \\ \ell^2\left(H^\perp\right) & \xrightarrow{M_v} & \ell^2\left(H^\perp\right) \end{array}$$

FIGURE 2.

i.e. $M_v \circ \lambda_{\omega,h} = \lambda_{\omega,h} \circ M_u$.

Proof. The proof follows directly from Proposition 4.3 □

5. Concrete Application: Signal Denoising via Gabor Multipliers on $\mathbb{R}^2$

To illustrate the practical utility of our theory, we consider a concrete example in signal processing. Let $G = \mathbb{R}^2$ be the time-frequency plane, which is a locally compact abelian group under vector addition. The dual group $\widehat{G}$ is isomorphic to $\mathbb{R}^2$.

5.1. **Setup of the Gabor System.** Let $H = \mathbb{Z}^2$ be a lattice subgroup of G. We choose a window function $g_0 \in L^2\left(\mathbb{R}^2\right)$, for instance, the Gaussian function:

$$g_0\left(t, \nu\right) = e^{-\pi(t^2+\nu^2)}.$$

We define a family of window functions indexed by H via translation:

$$g_{(m,n)} = L_{(m,n)} g_0, \quad (m, n) \in \mathbb{Z}^2,$$

where $L_{(m,n)}$ denotes the time-frequency shift operator. The family $\mathcal{G} = \left\{g_{(m,n)}\right\}_{(m,n)\in\mathbb{Z}^2}$ forms a Bessel admissible family (and in fact, a frame) for $L^2\left(\mathbb{R}^2\right)$ with appropriate lattice parameters.

According to Definition 3.1, the associated Bessel mapping $F : \mathbb{Z}^2 \to L^2\left(\mathbb{R}^2\right)$ is given by $F\left(m, n\right) = g_{(m,n)}$.

5.2. **The Multiplier Operator for Denoising.** Consider a noisy signal $f_{\text{noisy}} \in L^2\left(\mathbb{R}^2\right)$. The Gabor coefficients of this signal are given by:

$$c_{(m,n)} = \left\langle f_{\text{noisy}}, g_{(m,n)} \right\rangle.$$

In many applications (e.g., image denoising), the noise manifests as small coefficients, while the signal energy is concentrated in larger coefficients. We construct a **thresholding multiplier** $m : \mathbb{Z}^2 \to \mathbb{C}$ defined by:

$$m\left(m, n\right) = \begin{cases} 1 & \text{if } |c_{(m,n)}| \geq \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is a threshold parameter.

The generalized Bessel multiplier operator $M_{m,\mathcal{G}}$ acts on the noisy signal as follows:

$$M_{m,\mathcal{G}}(f_{\text{noisy}}) = \sum_{(m,n)\in\mathbb{Z}^2} m\left(m, n\right) c_{(m,n)} g_{(m,n)}.$$

This operation effectively removes (sets to zero) the coefficients below the threshold λ, thereby denoising the signal.

5.3. **Verification of Boundedness.** From Theorem 3.5, since $\mathcal{G}$ is a Bessel admissible family with bound B and m is a bounded sequence (since $m\left(m, n\right) \in \{0, 1\}$), the operator $M_{m,\mathcal{G}}$ is a bounded linear operator on $L^2\left(\mathbb{R}^2\right)$ with norm $\|M_{m,\mathcal{G}}\| \leq B \|m\|_\infty = B$.

This concrete example demonstrates that the abstract notion of a Bessel multiplier on an LCA group directly corresponds to a standard and widely used signal processing technique (Gabor thresholding), validating the relevance of our framework.

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References

1. G. Abbaspour, H. Hosseinnezhad and A. Rahimi, *Generalized Bessel Multiplier in Hilbert spaces*, Result. Math., 73 (2018), 85.
2. S.T. Ali, J.P. Antoine and J.P. Gazeau, *Coherent States, Wavelets, and Their Generalizations*, Springer-Verlag, 2000.
3. F. Arabyani-Neyshaburi, F. Esmaeelzadeh, R.A. Kamyabi Gol and F. Rashid, *Some new perspectives on the Generalized Bessel Multiplier*, Result. Math., 80 (2025), 231.
4. A.A. Arefijamal, R.A. Kamyabi Gol, R. Raisi and N. Tavallaei, *A New Approach to Continuous Riesz Bases*, J. Sci., Islam. Repub. Iran., 24 (2013), pp. 63-69.
5. P. Balazs, *Basic Definition and Properties of Bessel Multipliers*, J. Math. Anal. Appl., 325 (2007), pp. 571-585.
6. P. Balazs, D. Bayer and A. Rahimi, *Multiplier for continuous frame in Hilbert spaces*, J. Phys. A: Math. Theor., 45 (2012), 244023.
7. P. Balazs, B. Laback, G. Eckel and W. A. Deutsch, *Time-Frequency Sparsity by Removing Preceptually Irrelevant Components Using a Simple Model of Simultaneous Masking*, IEEE Trans. Audio Speech Lang. Process., 18 (2010), pp. 34-49.
8. M. Bownik and K. Ross, *The structure of translation-invariant spaces on locally compact abelian groups*, J. Fourier Anal. Appl., 21 (2015), pp. 849–884.
9. C. Cabrelli and V. Paternostro, *Shift-invariant Space on LCA groups*, J. Funct. Anal., 258 (2010), pp. 2034-2059.
10. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Am. Math. Soc., 72 (1952), pp. 341-366.
11. G.B. Folland, *A Course in Abstract Harmonic Analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.
12. J.-P. Gabardo and D. Han, *Frames Associated with Measurable Space*, Adv. Comput. Math., 18 (2003), pp. 127-147.
13. C. Heil, *A Basic Theory Primer*, Expanded ed., Birkhäuser, 2011.
14. E. Hewitt and A. Ross, *Abstract Harmonic Analysis*, Vol. I, Structure of Topological Groups, Springer-Verlag, New York, 1963.
15. E. Hewitt and A. Ross, *Abstract Harmonic Analysis*, Vol. II, Structure and Analysis for Compact Groups, Springer-Verlag, New York, 1970.
16. M.S. Jakobsen and J. Lemvig, *Co-compact Gabor systems on locally compact abelian groups*, J. Fourier Anal. Appl., 22 (2015), pp. 35-70.
17. M.S. Jakobsen and J. Lemvig, *Reproducing formulas for generalized translation invariant systems on locally compact abelian groups*, Trans. Am. Math. Soc., 368 (2016), pp. 8447-8480.

18. G. Kutyniok and D. Labate, *The Theory of reproducing on Locally Compact Abelian Groups*, Colloq. Math., 106 (2006), pp. 197-220.
19. P. Majdak and P. Balazs, W. Kreuzer and M. Dörfler, *Time-Frequency Method for Increasing the Signal-To-Noise Ratio in System Identification with Exponential Sweeps*, in: Proc. ICASSP, 2011.
20. G.J. Murphy, *C*-Algebras and Operator Theory*, Academic Press, 1990.
21. A. Pietsch, *Operator Ideals*, North-Holland Publishing Company, 1980.
22. A. Rahimi and P. Balazs, *Multipliers for p-Bessel sequences in Banach spaces*, Integral Equations Oper. Theory., 68 (2010), pp. 193-205.
23. S. Reiter and D. Stegeman, *Classical Harmonic Analysis and Locally Compact Groups*, Lond. Math. Soc. Monogr., 22 (2000).
24. R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer, 1960.
25. D. Wang and G.J. Brown, *Computational Auditory Scene Analysis: Principles, Algorithms, and Applications*, Wiley-IEEE Press, 2006.
26. K. Zhu, *Operator Theory in Function Spaces*, Mathematical Surveys and Monographs, 2007.

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